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# Generalized Relative Lower Order of Entire Function of Two Complex Variables.

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**Abstract:** In this paper we introduce the idea of the generalized relative order of entire function of two complex variables are discussed in this paper.

## 1. Introduction, Definition and Notations

Let $f$ and $g$ be any two entire functions defined in the complex plane $\mathbb{C}$ and $M_f(r) = max\{|f(z)| : |z| = r\}$, $M_g(r) = max\{|g(z)| : |z| = r\}$. In [6] defined the generalized order $\rho_f^{[l]}$ and generalized lower order $\lambda_f^{[l]}$ of an entire function $f$ for any integer $l \geq 2$ in the following way.

$$\rho_f^{[l]} = \lim_{r\to\infty} sup \frac{\log^{[l]} M_f(r)}{\log r}$$

and

$$\lambda_f^{[l]} = \lim_{r\to\infty} inf \frac{\log^{[l]} M_f(r)}{\log r}$$

where
$\log^{[k]} x = \log(\log^{k-1} x)$ , k =1,2,3……….. and $\log^{[0]} x = x$

when $l = 2$, the above definition coincides with the classical definition of order and lower order , which are as follows:

$$\rho_f = \lim_{r\to\infty} sup \frac{\log^{[2]} M_f(r)}{\log r}$$

and

$$\lambda_f = \lim_{r\to\infty} inf \frac{\log^{[2]} M_f(r)}{\log r}$$

Where $M_f(r)$ is strictly increasing and continuous.

In $[1\,,2]$ Bernal introduced the definition of relative order of $g$ with respect to $f$ , denoted by as follows :

$$\rho_g(f) = inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \ \forall\, r > r_0\,(\mu) > 0\}$$

$$= \lim_{r\to\infty} sup \frac{\log M_g^{-1}\ M_f(r)}{\log r}$$

Let $f(z_1\,, z_2)$ be a non-constant entire function of two complex variables $z_1$ and $z_2$ holomorphic in the closed polydisc

$$\{(z_1\,, z_2) : \ |z_j| \le r_j, \quad j = 1,2 \ \forall \ r_1 \ge 0\,, r_2 \ge 0\}$$

$$F(r_1, r_2) = max\{|f(z_1\,, z_2)| : |z_j| \le r_j, j = 1,2\}$$

Then by Hartogs theorem and maximum principle in $[4]$, $F(r_1, r_2)$ is an increasing function of $r_1, r_2$. The order $\rho = \rho(f)$ of $f(z_1\,, z_2)$ is defined in [3] as the infimum of all positive number μ for which

$$F(r_1, r_2) < exp[(r_1 r_2)^\mu]$$

holds for all sufficient large values of $r_1$ and $r_2$ another words

$$\rho_f = inf\{\mu > 0 : F(r_1, r_2) < exp[(r_1 r_2)^\mu]\ ;\ for\ all\ r_1 \ge R(\mu), r_2 \ge R(\mu)\}$$

Equivalent formula for $\rho_f$ in $[1\,,4]$ is

$$\rho(f) = \lim_{r_1, r_2\to\infty} sup \frac{\log\log F(r_1, r_2)}{\log(r_1 r_2)}$$

Let $h$ and $k$ be two functions defined on $\mathbb{R}$ such that $h\,, k : \ \mathbb{R} \to [-\infty, \infty]$. The order of $h$ relative to $k$ is

$$order(h : k) = inf[a > 0 : \ \exists\, C_a \in \ \mathbb{R}, \forall\, x \in \ \mathbb{R}, f(x) \le a^{-1} g(ax) + c_a\,]$$

If $H$ is an entire function then the growth function of $H$ is defined by

$$h(t) = sup[\log|H(z)|, |z| \le e^t], t \in \ \mathbb{R}$$

if $H$ and $K$ are two entire functions then the order of $H$ relative to $K$ is now defined by

$$order(H : K) = order(h : k)$$

as observed in [4], the expression $a^{-1}g(ax) + c_a$ may be replaced by $g(ax) + c_a$ if $g(t) = e^t$ because then the infimum in the case coincide.

Taking $c_a = 0$ in the above definition, are may easily verify that

$$order(H : K) = \rho_K(H)$$

i.e., the order $(H : K)$ coincides with Bernal's definition of relative order.

If $K = \exp z$ then order $(H : K)$ coincides with classical order of above papers by Kiselman and other.

Datta , Tanmay and Biswas, introduced the definition of relative order of an entire function $f(z_1 , z_2)$ with respect to an entire function $g(z_1 , z_2)$ as follows,

***Definition* 1.** Let $g(z_1 , z_2)$ be an entire function holomorphic in the closed polydisc $\{(z_1 , z_2) : |z_j| \leq r_j, j = 1,2\}$ and let

$$G(r_1, r_2) = max\{|g(z_1 , z_2)| : |z_j| \leq r_j, j = 1,2\}$$

$$\rho_g(f) = inf\{\mu > 0 : F(r_1, r_2) < G({r_1}^{\mu} , {r_2}^{\mu}); for\ r_1 \geq R(\mu), r_2 \geq R(\mu) \}$$

the definition coincides with that of classical (1) if $g(z_1, z_2) = exp\ z_1 . z_2$

In this paper we introduce the idea of relative order of entire functions of two complex variables.

***Definition* 2.** If $l \geq 1$ is positive integer, then the $l^{th}$ generalized relative order $f$ with respect to $g$, denoted by

$$\rho_g^{[l]}(f) = inf\left\{\mu > 0 : M_f(r_1, r_2) < M_g\left(exp^{[l-1]}({r_1}^{\mu}{r_2}^{\mu})\right); for\ all\ r_1 \geq R(\mu), r_2 \geq R(\mu)\right\}$$

$$= \lim_{r_1, r_2 \to \infty} \frac{\log^{[l]} M_g^{-1} M_f(r_1, r_2)}{\log r_1 r_2}$$

If $l = 1$ then $\rho_g^{[l]}(f) = \rho_g(f)$. If $l = 1$, $g(z_1, z_2) = e^{z_1 \cdot z_2}$, the classical order of $f$ $(c.f. [7])$

are can define the $l^{th}$ generalized relative lower order of $g$ with respect to $f$, denoted by $\lambda_f^{[l]}(g)$ as follows.

$$\lambda_f^{[l]}(g) = \lim_{r_1,r_2\to\infty} inf \frac{\log^{[l]} M_f^{-1} M_g(r_1,r_2)}{\log(r_1 r_2)}$$

***Definition* 3.** A non-constant, entire function $f$ is said to the property (R) if for any $\sigma > 1$ and for all sufficiently large $r_1, r_2$,

$\left[M_f(r_1,r_2)\right]^2 \le M_f({r_1}^{\sigma}, {r_2}^{\sigma})$ holds.

Our aim in this paper is to study some parallel basic properties of generalized relative lower order of entire functions (holomorphic functions or integral function) of two complex variables.

***Lemma* 1. [2]** Suppose $f$ is a non-constant entire function,

$\alpha > 1, 0 < \beta < \alpha, \delta > 1, 0 < \mu < \lambda$ and $n$ is a positive integer.

Then

$(i)$ $M_f(\alpha r_1, \alpha r_2) = \beta M_f(r_1,r_2)$

$(ii)$ $\exists\, k = k(\delta, f) > 0$ s.t

$$\left(M_f(r_1,r_2)\right)^{\delta} = k\left({r_1}^{\delta}, {r_2}^{\delta}\right) for\ r_1 > 0, r_2 > 0$$

$(iii)$ $\lim_{r_1,r_2\to\infty} \frac{M_f\left({r_1}^{\delta},{r_2}^{\delta}\right)}{M_f(r_1,r_2)} = \infty = \lim_{r_1,r_2\to\infty} \frac{M_f\left({r_1}^{\lambda},{r_2}^{\lambda}\right)}{M_f({r_1}^{\mu},{r_2}^{\mu})}$

$(iv)$ If $f$ is transcendental then

$$\lim_{r_1,r_2\to\infty} \frac{M_f\left({r_1}^{\delta},{r_2}^{\delta}\right)}{M_f(r_1,r_2)} = \infty = \lim_{r_1,r_2\to\infty} \frac{M_f\left({r_1}^{\lambda},{r_2}^{\lambda}\right)}{({r_1}^{n}{r_2}^{n})M_f({r_1}^{\mu},{r_2}^{\mu})}$$

***Lemma* 2. [2]** Let $f$ be an entire function satisfying the property (R), and let $\delta > 1$ and $n$ be a given positive integer. Then the inequality

$\left[M_f(r_1,r_2)\right]^n \leq M_f\left({r_1}^{\delta},{r_2}^{\delta}\right)$ holds for $r_1, r_2$ large enough.

***Lemma* 3**. Let $f, g$ *and* $h$ are any three entire functions. If

$M_g(r_1,r_2) \leq M_h(r_1,r_2)$ for all sufficiently large values of $r_1, r_2$ Then

$$\lambda_h^{[l]}(f) \leq \lambda_g^{[l]}(f) \ , \ where \ \ l \geq 1.$$

***Lemma* 4**. Every entire function $f$ satisfying the property (R) is transcendental.

***Lemma* 5**. [5 ] Let $f(z_1,z_2)$ be holomorphic in the polydisc $|z_j| = 2eR_j$ ; $R_j > 0, j = 1,2$ with $f(0) = 1$ and η be an arbitrary positive number not exceeding $\frac{3e}{2}$ then inside the polydisc but outside of a family encoding polydisc the sum of where radii is not greater than $4\,\eta\, R_1, R_2$ ,

we have

$$\log|f(z_1,z_2)| > -T(\eta)\ \log M_f\,(2eR_1, 2eR_2)$$

for $T(\eta) = 2 + \log\frac{3e}{2\eta}$

***Theorem* 1.** If $f_1, f_2, \ldots \ldots \ldots \ldots \ldots f_n \quad (n \geq 2)$ and $g$ are entire functions, then

$$\lambda_f^{[l]}(g) \geq \lambda_{f_i}^{[l]}(g)$$

where $l \geq 1$ , $f = f_1 \pm \sum_{k=2}^{n} f_k$ *and* $\lambda_{f_i}^{[l]}(g) = min\left\{\lambda_{f_k}^{[l]}(g)/k = 1,2, \ldots \ldots, n\right\}$

the equality holds when

$$\lambda_{f_i}^{[l]}(g) \neq \lambda_{f_k}^{[l]}(g) \,, \{k = 1,2, \ldots \ldots, n \ and\ k \neq i\}$$

***Proof*** : If $\lambda_f^{[l]}(g) = \infty$ then the result is obvious. So we suppose that $\lambda_f^{[l]}(g) < \infty$ we can clearly assume that $\lambda_{f_i}^{[l]}(g)$ is finite. By hypothesis $\lambda_{f_i}^{[l]}(g) \leq \lambda_{f_k}^{[l]}(g)$ for all $k = 1,2, \ldots \ldots i, \ldots \ldots \ldots \ldots \ldots, n$.we can suppose $\lambda_{f_i}^{[l]}(g) > 0$

now for any arbitrary $\varepsilon > 0$, we get for all sufficiently large values of $r_1, r_2$ that

$$M_{f_k}\left[exp^{[l-1]}(r_1)^{\lambda_{f_k}^{[l]}(g)-\varepsilon}(r_2)^{\lambda_{f_k}^{[l]}(g)-\varepsilon}\right] < M_g(r_1,r_2)$$

where $k = 1,2, \ldots \ldots, n$

i.e., $M_{f_k}(r_1, r_2) < M_g\left[\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_k}^{[l]}(g)-\varepsilon}}\right]$

where $k = 1,2, \dots \dots, n$

so, (1) $M_{f_k}(r_1, r_2) \leq M_g\left[\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_k}^{[l]}(g)-\varepsilon}}\right]$

where $k = 1,2, \dots \dots, n$

now for all sufficiently large values of $r_1, r_2$

$$M_f(r_1, r_2) < \sum_{k=1}^{n} M_{f_k}(r_1, r_2)$$

i.e., $M_f(r_1, r_2) < \sum_{k=1}^{n} M_g\left[\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_k}^{[l]}(g)-\varepsilon}}\right]$

i.e., (2) $M_f(r_1, r_2) < nM_g\left[\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)-\varepsilon}}\right]$

now in view of the first part of *Lemma* 1, we obtain from (2) for all sufficiently large values of $r_1, r_2$. that

$$M_f(r_1, r_2) < M_g\left[(n+1)\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)-\varepsilon}}\right]$$

i.e., $M_f\left[exp^{[l-1]}\left(\frac{r_1.r_2}{n+1}\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)-\varepsilon}}\right] < M_g(r_1, r_2)$

$$exp^{[l-1]}\left(\frac{r_1.r_2}{n+1}\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)-\varepsilon}} < M_f^{-1} M_g(r_1, r_2)$$

$$\lambda_{f_i}^{[l]}(g) - \varepsilon < \frac{\log^{[l]} M_f^{-1} M_g(r_1, r_2)}{\log r_1 . r_2 + o(1)}$$

$$\frac{\log^{[l]} M_f^{-1} M_g(r_1, r_2)}{\log r_1 . r_2 + o(1)} > \lambda_{f_i}^{[l]}(g) - \varepsilon$$

so

$$\lambda_f^{[l]}(g) = \lim_{r_1, r_2 \to \infty} inf \frac{\log^{[l]} M_f^{-1} M_g(r_1, r_2)}{\log r_1 . r_2 + O(1)} \geq \lambda_{f_i}^{[l]}(g) - \varepsilon$$

$\varepsilon > 0$ is arbitrary ,

(3) $\lambda_f^{[l]}(g) \geq \lambda_{f_i}^{[l]}(g)$

next let

$$\lambda_{f_i}^{[l]}(g) > \lambda_{f_k}^{[l]}(g) \ where\ k = 1,2, \ldots \ldots \ldots, n\ and\ k \neq i$$

as $\varepsilon > 0$ is arbitrary from the definition of generalized lower order of entire function of two complex variables



$$M_g(r_1, r_2) < M_f \left[ exp^{[l-1]} (r_1 r_2)^{\lambda_{f_i}^{[l]}(g) + \varepsilon} \right]$$

i.e., (4) $M_g \left[ \left( \log^{[l-1]}(r_1 . r_2) \right)^{\frac{1}{\lambda_{f_i}^{[l]}(g) + \varepsilon}} \right] < M_{f_i}(r_1, r_2)$

since , $\lambda_{f_i}^{[l]}(g) < \lambda_{f_k}^{[l]}(g)$ , $where\ k = 1,2, \ldots \ldots \ldots \ldots \ldots, n\ and\ k \neq i$

then in view of the third part of $Lemma$ 1 we obtain that.

(5) $\lim_{r_1, r_2 \to \infty} \frac{M_g \left[ \left( \log^{[l-1]}(r_1 . r_2) \right)^{\frac{1}{\lambda_{f_i}^{[l]}(g) + \varepsilon}} \right]}{M_g \left[ \left( \log^{[l-1]}(r_1 . r_2) \right)^{\frac{1}{\lambda_{f_i}^{[l]}(g) - \varepsilon}} \right]} = \infty, where\ k = 1,2, \ldots, n\ and\ k \neq i$

Therefore from (5) we obtain for all sufficiently large values of $r_1, r_2$ that

$$(6) \quad M_g\left[\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)+\varepsilon}}\right] > nM_g\left[\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)-\varepsilon}}\right]$$

For all $k \in \{1,2,\dots\dots n\}\backslash\{i\}$

Thus from (1), (4) & (6) we get for a sequence of values of $r_1, r_2$ tending to infinity that

$$M_{f_i}(r_1,r_2) > M_g\left[\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)+\varepsilon}}\right]$$

i.e., $M_{f_i}(r_1,r_2) > nM_g\left[\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)-\varepsilon}}\right]$

$$(7) \quad M_{f_i}(r_1,r_2) > nM_{f_k}(r_1,r_2) \ \forall\, k = 1,2,\dots\dots,n \ \ with\ k \neq i$$

so from (4) & (7) and in view of the first part of $Lemma$ 1 it follows for a sequence of values of $r_1, r_2$ tending to infinity that

$$M_f(r_1,r_2) \ \geq \ M_{f_i}(r_1,r_2) - \sum_{\substack{k=1\\k\neq i}}^{n} M_{f_k}(r_1,r_2)$$

i.e., $M_f(r_1,r_2) \ \geq \ M_{f_i}(r_1,r_2) - \frac{1}{n}\sum_{\substack{k=1\\k\neq i}}^{n} M_{f_i}(r_1,r_2)$

$$M_f(r_1,r_2) \ \geq \ M_{f_i}(r_1,r_2) - \left(\frac{n-1}{n}\right) M_{f_i}(r_1,r_2)$$

$$M_f(r_1,r_2) \ > \ \left(\frac{1}{n}\right) M_{f_i}(r_1,r_2)$$

$$M_f(r_1,r_2) \ > \ \left(\frac{1}{n}\right) M_g\left[\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)+\varepsilon}}\right]$$

i.e., $M_f(r_1,r_2) \ > \ M_g\left[\frac{\left(\log^{[l-1]}(r_1.r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)+\varepsilon}}}{n+1}\right]$

this gives for a sequence of values of $r_1, r_2$ tending to infinity that

$$M_f\left[exp^{l-1}\{(n+1)r_1r_2\}^{\lambda_{f_i}^{[l]}(g)+\varepsilon}\right] > M_g(r_1,r_2)$$

i.e., $\{(n+1)r_1r_2\}^{\lambda_{f_i}^{[l]}(g)+\varepsilon} > \log^{[l-1]} M_f^{-1}M_g(r_1,r_2)$

$$\lambda_{f_i}^{[l]}(g) + \varepsilon > \frac{\log^{[l]} M_f^{-1}M_g(r_1,r_2)}{\log r_1r_2 + 0(1)}$$

$$\lambda_{f_i}^{[l]}(g) \geq \lim_{r_1,r_2\to\infty} inf\,\frac{\log^{[l]} M_f^{-1}M_g(r_1,r_2)}{\log r_1r_2 + 0(1)}$$

(8) $\lambda_f^{[l]}(g) = \lim_{r_1,r_2\to\infty} inf\,\frac{\log^{[l]} M_f^{-1}M_g(r_1,r_2)}{\log r_1r_2} \leq \lambda_{f_i}(g)$

so from (3) & (8) we finally obtain that

$$\lambda_f^{[l]}(g) = \lambda_{f_i}^{[l]}(g)$$

whenever

$\lambda_{f_i}^{[l]}(g) \neq \lambda_{f_k}^{[l]}(g)\ \forall\, k \in \{1,2,\ldots\ldots n\}\backslash\{i\}$

***Theorem* 2.** Let $n,\ l$ be two positive integers with $n, l \geq 2$. Then

$$\frac{1}{n}\lambda_f^{[l]}(g) < \lambda_{f^n}^{[l]}(g) \leq \lambda_f^{[l]}(g).$$

***Proof***: From the first and second parts of $Lemma$ 1, we obtain that

$$\{M_f(r_1,r_2)\}^n \leq kM_f({r_1}^n,{r_2}^n) < M_f\left((k+1){r_1}^n,(k+1){r_2}^n\right),$$

(9) $n > 1$ and $r_1 > 0\,, r_2 > 0$

Where $k = k\,(n\,,f) > 0$. Therefore from (9) we obtain that

$$M_f^{-1}({r_1}^n,{r_2}^n) < \frac{\log^{[l]}\frac{1}{(k+1)}M_f^{-1}M_g({r_1}^n,{r_2}^n)}{\log {r_1}^n{r_2}^n}$$

i.e., (10) $\lambda_{f^n}^{[l]}(g) \geq \frac{1}{n}\lambda_f^{[l]}(g)$

on the other hand since $\{M_f(r_1, r_2)\}^n > M_f(r_1, r_2)$ for all sufficiently large values of $r_1, r_2$ . we have by $lemma$ 3

(11) $\lambda_{f^n}^{[l]}(g) \leq \lambda_f^{[l]}(g)$

thus the theorem follows$(10)$ $and$ $(11)$.

***Theorem* 3 .** Let $P$ be a polynomial if $f$ is transcendental then $\lambda_{Pf}^{[l]}(g) = \lambda_f^{[l]}(g)$, and if $g$ is transcendental then $\lambda_f^{[l]}(Pg) = \lambda_f^{[l]}(g)$. If $f$ $and$ $g$ are both transcendental then $\lambda_{Pf}^{[l]}(g) = \lambda_f^{[l]}(Pg) = \lambda_f^{[l]}(g) = \lambda_{Pf}^{[l]}(g) = \lambda_{Pf}^{[l]}(Pg)$. Hence $Pf$ $and$ $Pg$ denote the ordinary product of $P$ with $f$ $and$ $g$ respectively and $l \geq 1$.

***Proof* :** Let $m$ be the degree of $P(z)$. Then there exists $\alpha$ such that $0 < \alpha < 1$ and a positive integer $n(> m)$ for which

$$2\alpha \leq |P(z)| \leq (r_1 r_2)^n$$

holds on $|z_1| = r_1$ , $|z_2| = r_2$ for all sufficient large values of $r_1, r_2$ now by the first part of $Lemma$ 1 we obtain that $M_g\left(\frac{1}{\alpha}(\alpha r_1, \alpha r_2)\right) > \frac{1}{2\alpha} M_g(\alpha r_1, \alpha r_2)$,

i.e., (12) $M_g(\alpha r_1, \alpha r_2) < 2\alpha M_g(r_1, r_2)$

now let us consider $h(z) = P(z).f(z)$, then from equation (2) and in view of the fourth part of $Lemma$ 1 we get for any $s(> 1)$ and for all sufficiently large values of $r_1, r_2$, that

$$M_g(\alpha r_1, \alpha r_2) < 2\alpha M_g(r_1, r_2) \leq M_h(r_1, r_2) \leq (r_1 r_2)^n M_g(r_1, r_2) < M_g({r_1}^s, {r_2}^s)$$

So, $\lim_{r_1, r_2 \to \infty} \frac{\log^{[l]} M_f^{-1} M_g(\alpha r_1, \alpha r_2)}{\log r_1 r_2} \leq \lim_{r_1, r_2 \to \infty} inf \frac{\log^{[l]} M_f^{-1} M_h(r_1, r_2)}{\log r_1 r_2} \leq \lim_{r_1, r_2 \to \infty} inf \frac{\log^{[l]} M_f^{-1} M_g({r_1}^s, {r_2}^s)}{\log r_1 r_2}$

i.e.,

$$\lim_{r_1, r_2 \to \infty} inf \frac{\log^{[l]} M_f^{-1} M_g(\alpha r_1, \alpha r_2)}{\log r_1 r_2 + 0(1)} \leq \lim_{r_1, r_2 \to \infty} inf \frac{\log^{[l]} M_f^{-1} M_h(r_1, r_2)}{\log r_1 r_2} \leq \lim_{r_1, r_2 \to \infty} inf \frac{\log^{[l]} M_f^{-1} M_g({r_1}^s, {r_2}^s)}{\log {r_1}^s {r_2}^s} .S$$

and letting $S \to 1+$ we get.

(13) $\lambda_{Pf}^{[l]}(g) = \lambda_f^{[l]}(g)$

Similarly, when $g$ is transcendental are can easily prove that

(14) $\lambda_f^{[l]}(Pg) = \lambda_f^{[l]}(g)$

if $f$ and $g$ are both transcendental then the conclusion of the theorem can easily be obtained by combining (13) $and$ (14), and the theorem follows.

***Theorem* 4.** If $f_1, f_2, \ldots\ldots\ldots\ldots\ldots, f_n$ $(n \geq 2)$, $g$ are entire functions of two complex variables and $g$ has the property $(R)$, then

$$\lambda_f^{[l]}(g) \geq \lambda_{f_i}^{[l]}(g),$$

where $f = \prod_{k=1}^{n} f_k$ $and$ $\lambda_{f_i}^{[l]}(g) = min\left\{\lambda_{f_k}^{[l]}(g) \mid \text{k} = 1,2, \ldots\ldots\ldots, \text{n}\right\}$

The equality holds when $\lambda_{f_i}^{[l]}(g) \neq \lambda_{f_k}^{[l]}(g)$ $(k = 1,2, \ldots\ldots, n \; and \; k \neq i)$

Finally assume that $F_1$ $and$ $F_2$ are entire functions such that $f = \frac{F_1}{F_2}$ is also an entire function. Then $\lambda_f^{[l]}(g) = min\left\{\lambda_{F_1}^{[l]}(g), \lambda_{F_2}^{[l]}(g)\right\}$.

***Proof***: If $\lambda_f^{[l]}(g) = \infty$ then result is obvious by lemma 4. suppose that $\lambda_f^{[l]}(g) < \infty$ we can clearly assume that $\lambda_{f_i}^{[l]}(g)$ is finite. Also suppose that

$\lambda_{f_i}^{[l]}(g) \leq \lambda_{f_k}^{[l]}(g)$ where k = 1,2, ……… , n. we can suppose $\lambda_{f_i}^{[l]}(g) > 0$

let $\varepsilon > 0$, with $\varepsilon > \lambda_{f_i}^{[l]}(g)$, we have for all sufficiently large values of $r_1, r_2$ that

$$M_{f_k}\left[exp^{[l-1]}(r_1 r_2)^{\frac{1}{\lambda_{f_k}^{[l]}(g) - \varepsilon/2}}\right] < M_g(r_1, r_2), \text{ where k} = 1,2, \ldots\ldots\ldots, \text{n}.$$

i.e., $$M_f(r_1, r_2) < M_g\left[\log^{[l-1]}(r_1 r_2)^{\frac{1}{\lambda_{f_k}^{[l]}(g) - \varepsilon/2}}\right], \text{ where k} = 1,2, \ldots\ldots\ldots, \text{n}.$$

so, (15) $$M_{f_k}(r_1, r_2) \leq M_g\left[\log^{[l-1]}(r_1 r_2)^{\frac{1}{\lambda_{f_k}^{[l]}(g) - \varepsilon/2}}\right] \; for \; k = 1,2, \ldots\ldots, n$$

From (15) we have for all sufficiently large values of $r_1, r_2$ that

$$M_f(r_1,r_2) \ < \ \prod_{k=1}^{n} M_{f_k}(r_1,r_2)$$

i.e., $M_f(r_1,r_2) \ < \ \prod_{k=1}^{n} M_g\left[\log^{[l-1]}(r_1r_2)^{\frac{1}{\lambda_{f_k}^{[l]}(g)\,-\,\varepsilon/2}}\right]$

i.e., (16) $M_f(r_1,r_2) \ < \ \left[M_g\left[\log^{[l-1]}(r_1r_2)^{\frac{1}{\lambda_{f_k}^{[l]}(g)\,-\,\varepsilon/2}}\right]\right]^n$

observe that

(17) $\delta := \dfrac{\lambda_{f_k}^{[l]}(g) - \varepsilon/2}{\lambda_{f_i}^{[l]}(g) - \varepsilon} > 1$

Since $g$ has the property (R), in view of Lemma 2 and (17) we obtain from (16) for all sufficiently large values of $r_1, r_2$ that

$$M_f(r_1,r_2) \ < \ M_g \log^{[l-1]}(r_1r_2)^{\frac{\delta}{\lambda_{f_i}^{[l]}(g)\,-\,\varepsilon/2}} = \ M_g\left[\log^{[l-1]}(r_1r_2)^{\frac{1}{\lambda_{f_k}^{[l]}(g)\,-\,\varepsilon/2}}\right]$$

i.e., $M_f\left[exp^{[l-1]}(r_1r_2)^{\left(\lambda_{f_i}^{[l]}(g)\,-\,\varepsilon\right)}\right] \le M_g(r_1,r_2)$

$$(r_1r_2)^{\left(\lambda_{f_i}^{[l]}(g)\,-\,\varepsilon\right)} \ < \ \log^{[l-1]} M_f^{-1} M_g(r_1,r_2)$$

$$\left(\lambda_{f_i}^{[l]}(g) \ - \ \varepsilon\right)\log r_1r_2 \ < \log^{[l]} M_f^{-1} M_g(r_1,r_2)$$

$$\lambda_{f_i}^{[l]}(g) \ - \ \varepsilon \ < \ \frac{\log^{[l]} M_f^{-1} M_g(r_1,r_2)}{\log r_1r_2}$$

So, $\lambda_f^{[l]}(g) = \lim_{r_1,r_2\to\infty} inf \dfrac{\log^{[l]} M_f^{-1} M_g(r_1,r_2)}{\log r_1r_2} \ge \lambda_{f_i}^{[l]}(g) \ - \ \varepsilon$

$since\ \varepsilon \ > 0$ is arbitrary,

(18) $\lambda_f^{[l]}(g) \ \ge \ \lambda_{f_i}^{[l]}(g)$

Next , let

$$\lambda_{f_i}^{[l]}(g) \, < \, \lambda_{f_k}^{[l]}(g) \; where \; k = 1,2, \ldots \ldots \ldots .., n \; and \; k \neq l, Fix > 0$$

with

$$\varepsilon \, < \, \frac{1}{4} min \left\{ \lambda_{f_k}^{[l]}(g) - \lambda_{f_i}^{[l]}(g) : k \, \in \{1,2, \ldots ..., n\} \backslash \{i\} \right\}.$$

without loss of any generality we may assume that $f_k(0) = 1$

$$\text{where k} = 1,2, \ldots \ldots \ldots, \text{n and k} \neq \text{i}$$

Definition of generalized relative order of entire functions of two complex variables we obtain for a sequence of values of $R_1, R_2$ tending to infinity that

$$M_g(R_1, R_2) \, < \, M_{f_i}\left[exp^{[l-1]}(R_1 R_2)^{\lambda_{f_i}^{[l]}(g) + \varepsilon}\right]$$

i.e., (19) $\quad M_{f_i}(R_1, R_2) \, > \, M_g\left[\log^{[l-1]}(R_1 R_2)^{\frac{1}{\lambda_{f_i}^{[l]}(g) + \varepsilon}}\right]$

also for all sufficiently large values of $r_1, r_2$ we get

$M_{f_k}\left[exp^{[l-1]}(r_1, r_2)^{\lambda_{f_k}^{[l]}(g) - \varepsilon}\right] < \, M_g(r_1, r_2),$ where $\text{k} = 1,2, \ldots \ldots \ldots, \text{n}$ and $\text{k} \neq \text{i}$

i.e., $\quad M_{f_k}(r_1, r_2) \, < \, M_g\left[\log^{[l-1]}(r_1 r_2)^{\frac{1}{\lambda_{f_k}^{[l]}(g) - \varepsilon}}\right]$, where $\text{k} = 1,2, \ldots \ldots \ldots, \text{n}$ and $\text{k} \neq \text{i}$

since, $\quad \lambda_{f_i}^{[l]}(g) \, < \, \lambda_{f_k}^{[l]}(g)$

We get from above that,

(20) $\quad M_{f_k}(r_1, r_2) \, < \, M_g\left[\log^{[l-1]}(r_1 r_2)^{\frac{1}{\lambda_{f_i}^{[l]}(g) - \varepsilon}}\right]$

,where $\text{k} = 1,2, \ldots \ldots \ldots, \text{n}$ and $\text{k} \neq \text{i}$

now in view of $Lemma$ 5, taking $f_k(z_1, z_2)$ for $f(z_1, z_2), \eta = \frac{1}{16}$ , $2R_1 \; and \; 2R_2$ for $R_1, R_2$ it follows for the values of $z_1, z_2$ specified in the Lemma that

$$\log|f_k(z_1,z_2)| > -T(\eta)\log M_{f_k}\big(2e.2(R_1,R_2)\big),$$

$$\text{where } T(\eta) = 2+\log\left(\frac{3e}{2.\frac{1}{16}}\right) = 2+\log(24e)$$

i.e., $T(\eta) = 2+\log\left(\frac{3e}{\frac{1}{8}}\right) = 2+\log(24e)$

therefore

$$\log|f_k(z_1,z_2)| > -(2+\log(24e))\log M_{f_k}\big(4e(R_1,R_2)\big)$$

holds within and $|z_1| = 2R_1$ and $|z_2| = 2R_2$ but outside a family of excluded polydisc the sum of whose Raddi is not greater than

$$4.\frac{1}{16}\,.2(R_1,R_2) = \frac{1}{2}\,(R_1,R_2)$$

$$= \left(\frac{R_1}{2},\frac{R_2}{2}\right)$$

If, $r_1 \in (R_1,2R_1)\, and\, r_2 \in (R_2,2R_2)$

Then $|z_1| = r_1$ and $|z_2| = r_2$

(21) $\log|f_k(z_1,z_2)| > 7\,\log M_{f_k}\big(4e.(R_1,R_2)\big)$

Since $r_1 > R_1$ , $r_2 > R_2$

we have from above and (19) for a sequence of values of $r_1,r_2$ tending to infinite that

(22) $M_{f_i}(r_1,r_2) > M_{f_i}(R_1,R_2) > M_g\left[\log^{[l-1]}(R_1R_2)^{\frac{1}{\lambda^{[l]}_{f_i}(g)+\varepsilon}}\right] > M_g\left[\log^{[l-1]}\left(\frac{r_1r_2}{2}\right)^{\frac{1}{\lambda^{[l]}_{f_i}(g)+\varepsilon}}\right]$

Let $|z_{r_1}|$ and $|z_{r_2}|$ be a point on polydisc.

Such that

$M_{f_i}(r_1) = |f_i(r_1)|$ , $M_{f_i}(r_2) = |f_i(r_2)|$.

therefore as $r_1 > R_1$ and $r_2 > R_2$

from (20) & (21) , (22) it follows for a sequence of values of $r_1, r_2$ tending to infinite that

$$M_f\ (r_1, r_2) = max\{|f(z_1, z_2)| :\ |z_1| =\ r_1\ , |z_2| =\ r_2\}$$

$$= max\left\{\prod_{k=1}^{n} |f_k(z_1, z_2)| : |z_1| =\ r_1\ , |z_2| =\ r_2\right\}$$

so, $\quad M_f\ (r_1, r_2)\ \geq\ \prod_{\substack{k=1\\k\neq i}}^{n} |f_k(z_1, z_2)| \left|f_i\left(z_{r_1}\ , z_{r_2}\right)\right|$

i.e., $\quad M_f\ (r_1, r_2)\ \geq\ \prod_{\substack{k=1\\k\neq i}}^{n} \left[M_{f_k}(4eR_1\ , 4eR_2)\right]^{-7} M_g\left[\left(\log^{[l-1]}\left(\frac{r_1 r_2}{2}\right)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)\ +\ \varepsilon}}\right]$

$$\geq \prod_{\substack{k=1\\k\neq i}}^{n} M_g\left[\log^{[l-1]}(4eR_1\ , 4eR_2)^{\frac{1}{\lambda_{f_k}^{[l]}(g)\ -\ \varepsilon}}\right]^{-7} M_g\left[\left(\log^{[l-1]}\left(\frac{r_1 r_2}{2}\right)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)\ +\ \varepsilon}}\right]$$

$$= \prod_{\substack{k=1\\k\neq i}}^{n} M_g\left[\left(\log^{[l-1]}(4eR_1\ , 4eR_2)\right)^{\frac{1}{\lambda_{f_k}^{[l]}(g)\ -\ \varepsilon}}\right]^{-7} M_g\left[\left(\log^{[l-1]}\left(\frac{4er_1 r_2}{8e}\right)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)\ +\ \varepsilon}}\right]$$

Hence,

(23) $M_f\ (r_1, r_2) =\ \prod_{\substack{k=1\\k\neq i}}^{n} \left[M_g\left[\left(\log^{[l-1]}(4er_1 r_2)\right)^{\frac{1}{\lambda_{f_k}^{[l]}(g)\ -\ \varepsilon}}\right]\right]^{-7} \times M_g\left[\left(\log^{[l-1]}\left(\frac{4er_1 r_2}{8e}\right)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)\ +\ \varepsilon}}\right]$

on the other hands, we have

$\left(\log^{[l-1]}\left(\frac{4er_1 r_2}{8e}\right)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)\ +\ \varepsilon}}\ \geq\ \left(\log^{[l-1]}(4er_1 r_2\ )\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)\ +2\varepsilon}}$

asymptotically.

using Lemma 2 with $n = 2$

$$\delta = \frac{\lambda_{f_i}^{[l]}(g)\ + 3\varepsilon}{\lambda_{f_i}^{[l]}(g)\ + 2\varepsilon}\ > 1$$

$r_1, r_2$ tending to infinity

$$(24)\quad M_g\left[\left(\log^{[l-1]}\left(\frac{4er_1r_2}{8e}\right)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)+\varepsilon}}\right] \geq M_g\left[\left\{\left(\log^{[l-1]}(4er_1r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)+3\varepsilon}}\right\}\right]$$

Let, $L = min\left\{\lambda_k^{[l]}(g) : k \neq i\right\}$.

Choosing, $\delta = \frac{L-\varepsilon}{\lambda_{f_i}^{[l]}(g)+3\varepsilon} > 1$

Choose $\varepsilon > 0$

$$(25)\quad M_g\left[\left(\log^{[l-1]}(4er_1r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)+3\varepsilon}}\right] \geq$$

$$M_g\left[\left(\log^{[l-1]}(4er_1r_2)\right)^{\frac{1}{L-\varepsilon}}\right]^{7n} \prod_{\substack{k=1\\k\neq i}}^{n} M_g\left[\left[\left(\log^{[l-1]}(4er_1r_2)\right)^{\frac{1}{\lambda_{f_k}^{[l]}(g)-\varepsilon}}\right]\right]^{7}$$

for $r_1, r_2$ now from (23) , (24) & (25), it follows for a sequence of values of $r_1, r_2$ tending to infinity that

$$M_f\,(r_1,r_2) \geq M_g\left[\left(\log^{[l-1]}(4er_1r_2)\right)^{\frac{1}{\lambda_{f_i}^{[l]}(g)+3\varepsilon}}\right]$$

i.e., $M_f\left[exp^{[l-1]}(r_1r_2)^{\left(\lambda_{f_i}^{[l]}(g)+3\varepsilon\right)}\right] \geq M_g(4er_1r_2)$

$$(r_1r_2)^{\left(\lambda_{f_i}^{[l]}(g)+3\varepsilon\right)} \geq \log^{[l-1]} M_f^{-1}M_g(4er_1r_2)$$

$$\left(\lambda_{f_i}^{[l]}(g)+3\varepsilon\right)\log r_1r_2 \geq \log^{[l]} M_f^{-1}M_g(4er_1r_2)$$

$$\lambda_{f_i}^{[l]}(g)+3\varepsilon \geq \frac{\log^{[l]} M_f^{-1}M_g(4er_1r_2)}{\log(4er_1r_2)+0(1)}$$

If $\varepsilon \to 0$ then we get,

$$\lambda_{f_i}^{[l]}(g) \geq \lim_{r_1,r_2\to\infty} inf\,\frac{\log^{[l]} M_f^{-1}M_g(4er_1r_2)}{\log(4er_1r_2)+0(1)}$$

therefore

(26) $\lambda_f^{[l]}(g) = \lim_{r_1,r_2\to\infty} inf \frac{\log^{[l]} M_f^{-1} M_g(r)}{\log r_1 r_2} \leq \lambda_{f_i}^{[l]}(g)$

So from $(18)$ $and$ $(26)$ we finally obtain that

$$\lambda_f^{[l]}(g) = \lambda_{f_i}^{[l]}(g)$$

assume that

$$\lambda_{f_i}^{[l]}(g) \neq \lambda_{f_k}^{[l]}(g) \; \forall\, k \in \{1,2,\ldots\ldots,n\}\backslash\{i\}.$$

Let now

$f = \frac{F_1}{F_2}$ With $F_1, F_2, f$ entire and suppose

$$\lambda_{F_1}^{[l]}(g) \geq \lambda_{F_2}^{[l]}(g)$$

we have $F_1 = f.F_2$. thus

$$\lambda_{F_1}^{[l]}(g) = \lambda_F^{[l]}(g)$$

if

$$\lambda_F^{[l]}(g) < \lambda_{F_2}^{[l]}(g).$$

so it follows that

$$\lambda_{F_1}^{[l]}(g) < \lambda_{F_2}^{[l]}(g),$$

which contradicts

$$\lambda_{F_1}^{[l]}(g) \geq \lambda_{F_2}^{[l]}(g)$$

hence

$$\lambda_F^{[l]}(g) = \lambda_{\frac{F_1}{F_2}}^{[l]}(g) \geq \lambda_{F_2}^{[l]}(g)$$

$$= min\left\{\lambda_{F_1}^{[l]}(g), \lambda_{F_2}^{[l]}(g)\right\}$$

also suppose that

$$\lambda_{F_1}^{[l]}(g) > \lambda_{F_2}^{[l]}(g)$$

than

$$\lambda_{F_1}^{[l]}(g) = min\left\{\lambda_f^{[l]}(g), \lambda_{F_2}^{[l]}(g)\right\} = \lambda_{F_2}^{[l]}(g),$$

If, $\lambda_f^{[l]}(g) > \lambda_{F_2}^{[l]}(g)$

which is also a contradiction

thus, $\lambda_f^{[l]}(g) = \lambda_{\frac{F_1}{F_2}}^{[l]}(g) = min\left\{\lambda_{F_1}^{[l]}(g), \lambda_{F_2}{}^{[l]}(g)\right\}$.

This proves the theorem.

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